Subject Code: 24BM11RC04 R-24 Reg No: | | | | | | | | | | | |

GAYATRI VIDYA PARISHAD COLLEGE OF ENGINEERING FOR WOMEN

(Autonomous)

(Affiliated to Andhra University, Visakhapatnam)

II B.Tech. - I Semester Regular Examinations, Nov – 2025

COMPLEX AND FOURIER ANALYSIS

(ECE Branch)

1. All questions carry equal marks
2. Must answer all parts of the question at one place

Time: 3Hrs. **Max Marks: 70**

UNIT-I

1. a. Show that the function $u(x,y)=e^x\cos y$ is harmonic. Determine its harmonic conjugate $v(x,y)$ and find the analytic function $f(z)=u+iv$.

   b. Find the analytic function $f(z)=u+iv$, if $(u-v)=e^x(\cos y-\sin y)$.

OR

2. a. Evaluate by Cauchy's integral formula $\oint_C \frac{\sin^2 z}{(z-\frac{\pi}{6})^3}dz$ where $C:|z|=1$.

   b. Evaluate $\int_C \frac{e^{2z}}{z^2-3z+2}dz$ by Cauchy's integral formula, where 'C' is the circle $|z|=3$.

UNIT-II

3. a. Expand Cos z into a Taylor's series expansion about the point $z=\frac{\pi}{2}$.

   b. Expand $f(z)=\frac{1}{(z-1)(z-2)}$ in the region (i) $1<|z|<2$ (ii) $|z|>2$ (iii) $0<|z-1|<1$.

OR

4. a. Evaluate by Residue theorem, $\oint_C \frac{dz}{(z^2+1)(z^2-4)}$ where C is $|z|=1.5$.

   b. Show that, by Residue theorem, $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=\frac{2\pi}{\sqrt{3}}$.

UNIT-III

5. a. Find the Fourier series representing $f(x)=x, 0<x<2\pi$.

   b. Find the Fourier series to represent the function f(x) is given by

$$f(x)=\begin{cases}0, for -\pi<x<0\\ x^2, for\ 0<x<\pi\end{cases}$$

OR

6. a. Find the Half range Sine series for $f(x)=x(\pi-x), 0<x<\pi$.

   b. Obtain the Fourier series for the function $f(x)=x^2, in -\pi<x<\pi$.

UNIT-IV

7. a. Find the Fourier transform of $f(x) = \begin{cases} \cos x, & for\ |x| \le a \\ 0, & for\ |x| \ge a,\ a > 0 \end{cases}$

b. Find the Fourier sine and cosine transform of $f(x) = \frac{e^{-ax}}{x}$.

OR

8. a. Find the finite Fourier cosine transform of f(x) defined by $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}, where\ 0 < x < \pi$.

b. Evaluate the following using Parsavel's identity $\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx\ (a > 0)$.

UNIT-V

9. a. A tightly stretched string with fixed end points x=0 and x=l is initially in a position given by $y = y_0 sin^3 \frac{\pi x}{l}$, if it is released from rest from this position. Find the displacement y(x,t).

b. Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, where\ u(x,0) = 6e^{-3x}$ by the method of separation of variables.

OR

10. a. Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, where\ u(0,y) = 8e^{-3y}$ by the method of separation of variables.

b. Solve the one-dimensional heat flow equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ given that $u(0,t) = 0, u(l,t) = 0, t > 0, and\ u(x,0) = 3\sin\frac{\pi x}{l}, 0 < x < l$.

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Time: 3 Hrs Max. Marks: 70

UNIT-I

1. a. Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$.

Solution: Given $u(x, y) = e^x \cos y$.

Differentiating w.r.t x and y, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \text{ and } \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is a harmonic function.

Let $f(z) = u + iv$ is the analytic such that $u = e^x \cos y$.

Now $u_x = e^x \cos y$ and $u_y = -e^x \sin y$.

Replace x by z and y by 0, we get

$$u_x(z, 0) = e^z$$

$$u_y(z, 0) = 0$$

Since real part is given, by Milne-Thomson's method

$$\begin{aligned} f'(z) &= u_x(z, 0) - iu_y(z, 0) \\ &= e^z - i(0) \\ &= e^z. \end{aligned}$$

Integrating w.r.t. z, we get

$$\begin{aligned} f(z) &= \int e^z dz + c \\ &= e^z + c \end{aligned}$$

1. b. If $f(z) = u + iv$ is an analytic function of z and if $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.

Solution: $f(z) = u + iv$

$$if(z) = iu - v$$

Adding $(1 + i)f(z) = (u - v) + i(u + v)$.

Put $U = u - v$, $V = u + v$ and $F(z) = (1 + i)f(z)$.

$$\therefore F(z) = U + iV.$$

As $f(z)$ is analytic, $(1 + i)f(z)$ is analytic. Hence $F(z)$ is analytic.

Given $U = u - v = e^x(\cos y - \sin y)$.

$$\begin{aligned} U_x &= e^x(\cos y - \sin y) \\ U_y &= e^y(-\sin y - \cos y). \end{aligned}$$

Replacing x by z and y by 0, we get

$$\begin{aligned} U_x(z, 0) &= e^z(\cos 0 - \sin 0) = e^z \\ U_y(z, 0) &= e^z(-\sin 0 - \cos 0) = -e^z. \end{aligned}$$

Since real part is given, by Milne-Thomson's method

$$\begin{aligned} F'(z) &= U_x(z, 0) + iV_x(z, 0) \\ &= U_x(z, 0) - iU_y(z, 0) \\ &= e^z + ie^z = (1 + i)e^z. \end{aligned}$$

Integrating w.r.t. z, we get

$$\begin{aligned} \int F'(z)dz &= (1 + i) \int e^z dz + c \\ \Rightarrow F(z) &= (1 + i)e^z + c \\ \Rightarrow (1 + i)f(z) &= (1 + i)e^z + c \\ \Rightarrow f(z) &= e^z + \frac{c}{1 + i} \\ &= e^z + c', \quad c' = \frac{c}{1 + i} \\ \therefore f(z) &= e^z + c'. \end{aligned}$$

2. a. Evaluate by Cauchy's integral formula $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz$, where $C : |z| = 1$.

Solution: Let $I = \oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz$

C is the circle $|z| = 1$ with centre at $(0,0)$ and a radius of 1.

The singularities are given by $(z-\pi/6)^3 = 0 \Rightarrow z = \pi/6$.

For $z = \frac{\pi}{6}$, $|z| = \left|\frac{\pi}{6}\right| = 0.52 < 1$

Hence $z = \frac{\pi}{6}$ lies inside C.

Let $f(z) = \sin^2 z$. Then $f(z)$ is analytic inside and on C.

$f'(z) = 2\sin z\cos z = \sin 2z$, $f''(z) = 2\cos 2z$.

By Cauchy's integral formula for derivative,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)$$

$$\Rightarrow \oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz = \frac{2\pi i}{2!}f''\left(\frac{\pi}{6}\right)$$

$$= \pi i f''\left(\frac{\pi}{6}\right)$$

$$= \pi i\left[2\cos 2z\right]_{z=\frac{\pi}{6}}$$

$$= 2\pi i\left[\cos\frac{\pi}{3}\right]$$

$$= \pi i.$$

2. b. Evaluate $\oint_C \frac{e^{2z}}{z^2-3z+2}dz$ by Cauchy's integral formula, where C is the circle $|z| = 3$.

Solution: Let $I = \oint_C \frac{e^{2z}}{(z-1)(z-2)}dz$, where C is the circle $|z| = 3$.

The singular points are given by $(z-1)(z-2) = 0 \Rightarrow z = 1, z = 2$.

C is the circle $|z| = 3$ with centre at $(0,0)$ and a radius of 3.

If $z = 1$, then $|z| = |1| = 1 < 3$.

If $z = 2$, then $|z| = |2| = 2 < 3$.

So $z = 1$ and $z = 2$ are singular points inside C.

Let $f(z) = e^{2z}$. Therefore, $f(z)$ is analytic inside and on C.

Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$\Rightarrow A(z-2) + B(z-1) = 1$$

Put $z = 1$, then $-A = 1 \Rightarrow A = -1$

Put $z = 2$, then $B = 1$.

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

By Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-1)(z-2)} = \oint_C \frac{f(z)}{z-2} dz - \oint_C \frac{f(z)}{z-1} dz$$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z-1)(z-2)} = 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i \left(e^4 - e^2\right).$$

UNIT-II

3. a. Expand $\cos z$ into a Taylor's series expansion about the point $z = \pi/2$.

Solution: Let $f(z) = \cos z$.

The Taylor's series for $f(z)$ about $z = \pi/2$ is

$$f(z) = f(\pi/2) + \frac{(z-\pi/2)}{1!} f'(\pi/2) + \frac{(z-\pi/2)^2}{2!} f''(\pi/2) + \frac{(z-\pi/2)^3}{3!} f'''(\pi/2) + \cdots$$

Now $f(z) = \cos z$. Hence $f(\pi/2) = 0$

$f'(z) = -\sin z$. Hence $f'(\pi/2) = -1$

$f''(z) = -\cos z$. Hence $f''(\pi/2) = 0$

$f'''(z) = \sin z$. Hence $f'''(\pi/2) = 1$

$\therefore$ The Taylor's series for $\cos z$ about $z = \pi/2$ is

$$\cos z = -\frac{(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

The expansion is valid throughout the complex plane.

Alter: To expand $f(z)$ about $z = \pi/2$, put $z - \pi/2 = w \Rightarrow z = w + \pi/2$.

$$\begin{aligned}
\cos z &= \cos\left(w + \frac{\pi}{2}\right) \\
&= \cos w \cos\frac{\pi}{2} - \sin w \sin\frac{\pi}{2} \\
&= 0 - \sin w \\
&= -\left[w - \frac{w^3}{3!} + \frac{w^5}{5!} + \cdots\right] \\
&= -w + \frac{w^3}{3!} - \frac{w^5}{5!} \\
&= -\frac{(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots
\end{aligned}$$

3. b. Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the regions: (i) $1 < |z| < 2$ (ii) $|z| > 2$ (iii) $0 < |z-1| < 1$.

Solution: Here $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

$f(z)$ is not analytic at $z = 2$ and $z = 1$.

(i). $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$.

So, we expand as Laurent's series about $z = 0$.

Now $1 < |z| \Rightarrow \left|\dfrac{1}{z}\right| < 1$ and $|z| < 2 \Rightarrow \left|\dfrac{z}{2}\right| < 1$.

So, the Laurent's series is

$$\begin{aligned}
f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\
&= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right] \\
&= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right] - \left[z^{-1} + z^{-2} + z^{-3} + \cdots\right] \\
&= \cdots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots
\end{aligned}$$

(ii) Given $|z| > 2$.

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$.

So, the expansion is Laurent's series about $z = 0$.

Now $|z| > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < \frac{1}{2} < 1$.

$\therefore$ The Laurent's series is

$$\begin{aligned}
\therefore f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\
&= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] \\
&= \left[z^{-1}+2z^{-2}+4z^{-3}+\cdots\right] - \left[z^{-1}+z^{-2}+z^{-3}+\cdots\right] \\
&= \cdots + 7z^{-4} + 3z^{-3} + z^{-2} + \cdots
\end{aligned}$$

(iii). Given $0 < |z-1| < 1$.

$f(z)$ is analytic in the region $|z-1| < 1$ about $z = 1$.

So the expansion of $f(z)$ in the region is Laurent's series about $z = 1$.

Put $w = z - 1 \Rightarrow z = w + 1$. $\therefore 0 < |w| < 1$.

$\therefore$ The Laurent's series is

$$\begin{aligned}
f(z) = -\frac{1}{z-1} + \frac{1}{z-2} &= -\frac{1}{w+1-1} + \frac{1}{w+1-2} \\
&= -\frac{1}{w} + \frac{1}{w-1} \\
&= -\frac{1}{w} - \frac{1}{1-w} \\
&= -\frac{1}{w} - (1-w)^{-1} \\
&= -\frac{1}{w} - \sum_{n=0}^{\infty} w^n \\
&= -(z-1)^{-1} - \sum_{n=0}^{\infty} (z-1)^{-1} \\
&= -(z-1)^{-1} - \left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right] \\
&= -\left[\frac{1}{z-1} + 1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right] \\
&= -\sum_{n=-1}^{\infty} (z-1)^n \text{ if } 0 < |z-1| < 1.
\end{aligned}$$

4. a. Evaluate by Residue theorem, $\oint\limits_C \frac{dz}{(z^2+1)(z^2-4)}dz$, where C is $|z| = 1.5$.

Solution: Let $f(z) = \frac{1}{(z^2+1)(z^2-4)}$.

The poles are given by $(z^2+1)(z^2-4) = 0 \Rightarrow z = \pm i, 2, -2$.

C is the circle $|z| = 1.5$ with centre at $(0,0)$ and a radius of 1.5.

For $z = i$, $|z| = |i| = 1 < 1.5$. Hence $z = i$ lies inside C.

For $z = -i$, $|z| = |-i| = 1 < 1.5$. Hence $z = -i$ lies inside C.

For $z = 2$, $|z| = |2| = 2 > 1.5$. Hence $z = 2$ lies outside C.

For $z = -2$, $|z| = |-2| = 2 > 1.5$. Hence $z = -2$ lies outside C.

$z = i$ is a simple pole.

$$\begin{aligned}
\text{Res}\{f(z); i\} &= \lim_{z\to i}\Big[(z-i)f(z)\Big] \\
&= \lim_{z\to i}\left[(z-i)\frac{1}{(z+i)(z-i)(z^2-4)}\right] \\
&= \frac{1}{2i(-1-4)} \\
&= -\frac{1}{10i}.
\end{aligned}$$

$z = -i$ is a simple pole.

$$\begin{aligned}
\text{Res}\{f(z); -i\} &= \lim_{z\to -i}\Big[(z+i)f(z)\Big] \\
&= \lim_{z\to -i}\left[(z+i)\frac{1}{(z+i)(z-i)(z^2-4)}\right] \\
&= \frac{-1}{2i(-1-4)} \\
&= \frac{1}{10i}.
\end{aligned}$$

Hence by Cauchy's residue theorem,

$$\oint\limits_C f(z)dz = 2\pi i \times \big(\text{sum of the residues at the poles within } C\big)$$

$$\Rightarrow \oint\limits_C \frac{dz}{(z^2+1)(z^2-4)}dz = 2\pi i\left[-\frac{1}{10i} + \frac{1}{10i}\right] = 0.$$

4. b. Show that by Residue theorem $\int\limits_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.
To evaluate this, consider the unit circle $|z| = 1$ as contour C.
Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$.

$$
\begin{aligned}
\therefore I &= \oint_C \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz} \\
&= \frac{2}{i} \oint_C \frac{dz}{z^2+4z+1} \\
&= \frac{2}{i} \oint_C f(z)dz
\end{aligned}
$$

where $f(z) = \frac{1}{z^2+4z+1}$.
The poles of $f(z)$ are given by $z^2 + 4z + 1 = 0$

$$
\begin{aligned}
\Rightarrow z &= \frac{-4 \pm \sqrt{16-4}}{2} \\
&= \frac{-4 \pm 2\sqrt{3}}{2} \\
&= -2 \pm \sqrt{3} \\
&= -2+\sqrt{3} \text{ or } -2-\sqrt{3} \text{ which are simple poles.}
\end{aligned}
$$

Now $z^2 + 4z + 1 = \left[z - (-2+\sqrt{3})\right]\left[z - (-2-\sqrt{3})\right]$.

The contour is $|z| = 1$

If $z = -2+\sqrt{3}$, then $|z| = |2+\sqrt{3}| = -2+\sqrt{3} < 1$.

$\therefore z = -2+\sqrt{3}$ lies inside C.

If $z = -2-\sqrt{3}$, then $|z| = |-2-\sqrt{3}| = 2+\sqrt{3} < 1$.

$\therefore z = -2-\sqrt{3}$ lies outside C.

$$
\begin{aligned}
\text{Res}\left\{f(z); -2+\sqrt{3}\right\} &= \lim_{z\to -2+\sqrt{3}} \left[z - (-2+\sqrt{3})\right] f(z) \\
&= \lim_{z\to -2+\sqrt{3}} \left[z - (-2+\sqrt{3})\right]\left[\frac{1}{\left[z-(-2+\sqrt{3})\right]\left[z-(-2-\sqrt{3})\right]}\right] \\
&= \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} \\
&= \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}
$$

$\therefore$ By Cauchy's residue theorem,

$$\oint_C f(z)dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at poles lies inside } C$$
$$= 2\pi i \times \frac{1}{2\sqrt{3}}$$
$$= \frac{\pi i}{\sqrt{3}}.$$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\oint_C f(z)dz = \frac{2}{i} \times \frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

UNIT-III

5. a. Find the Fourier series representing $f(x) = x, 0 < x < 2\pi$.

Solution: Given $f(x) = x$ in $(0, 2\pi)$.

The Fourier series expansion of $f(x)$ in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \cdots\cdots \quad (1)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = \frac{1}{2\pi}\left[4\pi^2 - 0\right] = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx$$
$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{2\pi}$$
$$= \frac{1}{\pi n^2}\left[\cos 2n\pi - \cos 0\right]$$
$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx$$
$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$
$$= -\frac{1}{n\pi}\left[x\cos nx\right]_0^{2\pi}$$
$$= -\frac{1}{n\pi}\left[2\pi\cos 2n\pi\right]$$
$$= -\frac{2}{n}$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = \pi + \sum_{n=1}^{\infty}(0)\cos nx + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin nx$$
$$x^2 = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

5. b. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$

Solution: Given $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$

The Fourier series expansion in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad \cdots\cdots \quad (1)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)dx + \int_{0}^{\pi} x^2 dx\right]$$
$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{\pi^3}{3}\right]$$
$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx dx + \int_{0}^{\pi} x^2 \cos nx dx\right]$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$
$$= \frac{k}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right]$$
$$= \frac{2\cos n\pi}{n^2}$$
$$= \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx dx + \int_{0}^{\pi} x^2 \sin nx dx\right]$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx dx$$
$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2\cos nx}{n^3}\right]$$
$$= \frac{1}{\pi}\left[\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left(0 + \frac{2\cos 0}{n^3}\right)\right]$$
$$= \frac{1}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$
$$= \frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}[(-1)^n - 1]$$

Substitute these values of a_0, a_n and b_n in (1), we get

$$f(x) = \frac{\pi^2/3}{2} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left[\frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}[(-1)^n - 1]\right]\sin nx$$
$$= \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left[\frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}[(-1)^n - 1]\right]\sin nx$$

6. a. Find the half range sine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Solution: Given $f(x) = x(\pi - x)$ in $0 < x < \pi$.

The Fourier sine series expansion of $f(x)$ in $(0, \pi)$ is

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \cdots\cdots \quad (1)$$

where

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \\
&= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx dx \\
&= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{-2}{\pi}\left[(\pi x - x^2)\left(\frac{\cos nx}{n}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{-2}{\pi}\left[\left(0 + 2\frac{\cos n\pi}{n^3}\right) - \left(0 + 2\frac{\cos 0}{n^3}\right)\right] \\
&= \frac{-2}{\pi}\left[\frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \\
&= \frac{4}{\pi n^3}\left[1 - (-1)^n\right] \\
&= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8}{\pi n^3}, & \text{when } n \text{ is odd} \end{cases}
\end{aligned}
$$

Substitute b_n in (1), we get

$$
\begin{aligned}
x(\pi - x) &= \sum_{n=1,3,5,\cdots} \frac{8}{\pi n^3}\sin nx \\
&= \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)
\end{aligned}
$$

6. b. Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$.

Solution: Given $f(x) = x^2, -\pi < x < \pi$.

For the given function, $f(-x) = (-x)^2 = x^2 = f(x)$. Hence the given $f(x)$ is an *even function* in the interval $(-\pi, \pi)$. Therefore, all of $b_n = 0$ and the Fourier series for $f(x)$ over $(-\pi, \pi)$ is

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \cdots\cdots \quad (1)
$$

where

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2}{3}\pi^2$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_0^\pi x^2\cos nx dx \\
&= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi \\
&= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^\pi \\
&= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right] \\
&= \frac{4}{n^2}\cos n\pi \\
&= \frac{4}{n^2}(-1)^n
\end{aligned}$$

Substituting the values of a_0 and a_n in (1), we get

$$\begin{aligned}
x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx \\
&= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)
\end{aligned}$$

UNIT-IV

7. a Find the Fourier transform of $f(x) = \begin{cases} \cos x, & \text{for } |x \leq a \\ 0, & \text{for } |x| \geq a, a > 0 \end{cases}$.

Solution: The Fourier transform of $f(x)$ is

$$\begin{aligned}
F\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \int_{-\infty}^{-a} 0\cdot e^{isx}dx + \int_{-a}^{a} \cos xe^{isx}dx + \int_{a}^{\infty} 0\cdot e^{isx}dx \\
&= \int_{-a}^{a} \cos xe^{isx}dx \\
&= \left[\frac{e^{isx}}{1-s^2}(is\cos x + \sin x)\right]_{-a}^{a} \\
&= \left[\frac{e^{isa}}{1-s^2}(is\cos a + \sin a) - \frac{e^{-isa}}{1-s^2}(is\cos a - \sin a)\right] \\
&= 2i\frac{is\cos a}{1-s^2}\left[\frac{e^{isa} - e^{-isa}}{2i}\right] + \left[\frac{e^{isa} + e^{-isa}}{2}\right]\frac{2\sin a}{1-s^2} \\
&= -\frac{2s\cos a}{1-s^2}\sin as + \frac{2\sin a}{1-s^2}\cos as \\
&= \frac{2}{1-s^2}\left[\sin a\cos as - s\cos a\sin as\right].
\end{aligned}$$

7. b. Find the Fourier sine and cosine transform of $f(x) = \dfrac{e^{-ax}}{x}$.

Solution: Let $f(x) = \dfrac{e^{-ax}}{x}$, $a > 0$.

The Fourier Sine transform of $f(x)$ is given by

$$\begin{aligned}
F_S\{f(x)\} &= \int_{0}^{\infty} f(x)\sin sx dx \\
&= \int_{0}^{\infty} \frac{e^{-ax}}{x}\sin sx dx = I, \text{ say} \qquad \cdots\cdots \quad (1)
\end{aligned}$$

Differentiating (1) w.r.t s, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{x}(x\cos sx)\,dx$$
$$= \int_0^\infty e^{-ax}\cos sx\,dx$$
$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$
$$= \frac{a}{s^2+a^2}$$

Integrating, we have

$$I = \int \frac{a}{s^2+a^2}ds$$
$$= \tan^{-1}\left(\frac{s}{a}\right) + A \qquad \cdots\cdots \quad (2)$$

Putting $s = 0$ in (1), we get

$$I = \int_0^\infty \frac{e^{-ax}}{x}\sin 0 dx = 0 \qquad \cdots\cdots \quad (3)$$

Putting $s = 0$ in (2), we get $I = 0 + A = A$ $\cdots\cdots$ (4)

From (3) and (4), we have $A = 0$. Hence

$$I = F_S\left(\frac{e^{-ax}}{x}\right) = \tan^{-1}\left(\frac{s}{a}\right).$$

The Fourier Cosine transform of $f(x)$ is given by

$$F_S\{f(x)\} = \int_0^\infty f(x)\cos sx dx$$
$$= \int_0^\infty \frac{e^{-ax}}{x}\cos sx dx = I, \text{ say} \qquad \cdots\cdots \quad (1)$$

Differentiating (1) w.r.t s, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = -\int_0^\infty \frac{e^{-ax}}{x}(x \sin sx)\,dx$$
$$= -\int_0^\infty e^{-ax} \sin sx dx$$
$$= -\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$
$$= -\frac{s}{s^2+a^2}$$

Integrating, we have

$$I = -\int \frac{s}{s^2+a^2}ds$$
$$= -\frac{1}{2}\log(s^2+a^2) + A$$

8. a. Find the finite Fourier Cosine transform of $f(x)$ defined by $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$, where $0 < x < \pi$.

Solution: Given $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$, where $0 < x < \pi$ and $l = \pi$.

We have

$$F_C(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}dx$$
$$= \int_0^\pi f(x)\cos nx dx$$
$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-1+\frac{x}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right) + \left(\frac{1}{\pi}\right)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$
$$= \left[\left(-1+\frac{x}{\pi}\right)\frac{\cos nx}{n^2}\right]_0^\pi$$
$$= \left[(-1+1)\frac{\cos n\pi}{n^2} - (-1+0)\frac{\cos 0}{n^2}\right]$$
$$= \frac{1}{n^2},\ n > 0$$

and if $n = 0$,

$$F_C(n) = \int_0^l f(x)dx$$
$$= \int_0^\pi \left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)dx$$
$$= \left[\frac{\pi x}{3} - \frac{x^2}{2} + \frac{x^3}{6\pi}\right]_0^\pi$$
$$= \frac{\pi^2}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{6\pi}$$
$$= 0.$$

8. b. Evaluate the following using Parseval's identity: $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx (a > 0)$.

Solution: (i) Let $f(x) = g(x) = e^{-ax}, a > 0$. Then

$$F_S(s) = G_S(s) = \frac{s}{s^2+a^2}.$$

Using Parseval's identity for Fourier sine transforms,

$$\frac{2}{\pi}\int_0^\infty |F_S(s)|^2 ds = \int_0^\infty |f(x)|^2 dx$$
$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds = \int_0^\infty |e^{-ax}|^2 dx$$
$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \int_0^\infty e^{-2ax} dx$$
$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty = 0 - \frac{1}{-2a} = \frac{1}{2a}$$
$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \frac{\pi}{4a}$$
$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

UNIT-V

9. a. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3(\pi x/l)$. If it is released from rest from this position. Find the displacement of $y(x,t)$.

Solution: The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \cdots\cdots \quad (1)$$

subject to

$$y(0,t) = 0, \quad \text{for all } t \quad \cdots\cdots \quad (2)$$

$$y(l,t) = 0, \quad \text{for all } t \quad \cdots\cdots \quad (3)$$

$$y(x,0) = y_0 \sin^3(\pi x/l), \quad 0 \le x \le l \ \cdots \quad (4)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0, \quad 0 \le x \le l \ \cdots\cdots \quad (5)$$

The required solution of (1) is given by

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt).$$

Using conditions (2) and (3), we have

$$y(0,t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0 \Rightarrow c_1 = 0.$$

$$y(l,t) = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\Rightarrow \sin pl = 0$$

$$\Rightarrow pl = n\pi, \quad n = 1, 2, \cdots$$

$$\Rightarrow p = \frac{n\pi}{l}, \quad n = 1, 2, \cdots$$

$\therefore$ The general solution of (1) satisfying (2) and (3) is

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \cdots\cdots \quad (6)$$

where a_n and b_n are constants to be determined using (4) and (5).

Differentiating (6) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(- a_n \frac{n\pi c}{l} \sin \frac{n\pi ct}{l} + b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}.$$

Using condition (5) i.e., $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$, we get

$$0 = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} \Rightarrow b_n = 0 \quad \left[\because \sin\frac{n\pi x}{l} \neq 0\right]$$

Substituting $b_n = 0$ in (6), we get

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}. \quad \cdots\cdots \quad (7)$$

Using condition (4) i.e., $y(x,0) = y_0 \sin^3(\pi x/l)$ in (7), we get

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = y_0 \sin^3(\pi x/l), \; 0 \leq x \leq l.$$

We have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

$$\therefore \quad \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right].$$

Comparing the coefficients of like terms $a_1 = \frac{3y_0}{4}, a_3 = \frac{-y_0}{4}$ and $a_2 = a_4 = a_5 = \cdots = 0$.

Hence required solution is

$$\begin{aligned} y(x,t) &= \frac{3y_0}{4}\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \frac{y_0}{4}\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l} \\ &= \frac{y_0}{4}\left[3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right]. \end{aligned}$$

9. b. Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 6e^{-3x}$ by the method of separation of variables.

Solution: Here u is a function of x and t.

Let $\quad u = X(x)\cdot T(t) = XT \quad \cdots\cdots \quad (1)$

where X is a function of x only and T is a function of t only, be a solution of the given equation.

Then $\frac{\partial u}{\partial x} = X'T$, $\frac{\partial u}{\partial t} = XT'$.

Substituting in the given equation, we have

$$X'T = 2XT' + XT \Rightarrow X'T = (2T' + T)X.$$

Separating the variables, we get

$$\frac{X'}{X} = \frac{2T' + T}{T}. \qquad \cdots\cdots \quad (2)$$

Since x and t are independent variables, equation (2) can hold only when each side is equal to same constant, say k.

$$\therefore \frac{X'}{X} = k \Rightarrow \frac{dX}{X} = kdx \Rightarrow \log X = kx + \log c_1 \Rightarrow \log \frac{X}{c_1} = kx \Rightarrow X = c_1 e^{kx} \quad \cdots\cdots \quad (3)$$

and

$$\frac{2T' + T}{T} = k \Rightarrow \frac{2T'}{T} + 1 = k \Rightarrow \frac{T'}{T} = \frac{k-1}{2}$$

$$\Rightarrow \frac{dT}{T} = \left(\frac{k-1}{2}\right) dt$$

$$\Rightarrow \log T = \left(\frac{k-1}{2}\right) t + \log c_2$$

$$\Rightarrow \log \frac{T}{c_2} = \left(\frac{k-1}{2}\right) t$$

$$\Rightarrow T = c_2 e^{\frac{1}{2}(k-1)t}. \qquad \cdots\cdots \quad (4)$$

From (1), (3) and (4), we have

$$u = u(x,t) = c_1 e^{kx} \cdot c_2 e^{\frac{1}{2}(k-1)t} = A e^{kx} e^{\left(\frac{k-1}{2}\right)t}, \text{ where } A = c_1 c_2.$$

Since $u(x,0) = 6e^{-3x} \Rightarrow Ae^{kx} = 6e^{-3x} \Rightarrow A = 6$ and $k = -3$.

$\therefore$ The unique solution of the given equation is

$$u = 6e^{-3x} \cdot e^{-2t} = 6e^{-(3x+2t)}.$$

10. a. Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, where $u(0,y) = 8e^{-3y}$ by the method of separation of variables.

Solution: Here u is a function of x and y.

Let $\quad u(x,y) = X(x) \cdot Y(y) = XY \quad \cdots\cdots \quad (1)$

where X is a function of x only and Y is a function of y only, be a solution of the given equation.

Then $\frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$.

Substituting these values in the given equation, we have

$$X'Y = 4XY' \Rightarrow \frac{X'}{X} = 4\frac{Y'}{Y}. \qquad \cdots\cdots \quad (2)$$

Since x and y are independent variables, equation (2) can hold only when each side is equal to same constant, say k.

$$\therefore \frac{X'}{X} = k \Rightarrow \frac{dX}{X} = kdx \Rightarrow \log X = kx + \log c_1 \Rightarrow \log\frac{X}{c_1} = kx \Rightarrow X = c_1e^{kx} \quad \cdots\cdots \quad (3)$$

and

$$4\frac{Y'}{Y} = k \Rightarrow \frac{Y'}{Y} = \frac{k}{4} \Rightarrow \frac{dY}{Y} = \frac{k}{4}dy \Rightarrow \log Y = \frac{ky}{4} + \log c_2 \Rightarrow \log\frac{Y}{c_2} = \frac{ky}{4} \Rightarrow Y = c_2e^{\frac{ky}{4}} \cdots \quad (4)$$

From (1), (3) and (4), we have

$$u = u(x,y) = c_1e^{kx} \cdot c_2e^{\frac{ky}{4}} = Ae^{k(x+\frac{y}{4})}, \text{ where } A = c_1c_2.$$

Since $u(0,y) = 8e^{-3y} \Rightarrow Ae^{\frac{ky}{4}} = 8e^{-3y} \Rightarrow A = 8$ and $k = -12$.

$\therefore$ The unique solution of the given equation is

$$u(x,y) = 8e^{-12(x+\frac{y}{4})} = 8e^{-12x-3y}.$$

10. b. Solve one-dimensional heat flow equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ given that $u(0,t) = 0$ and $u(l,t) = 0, t > 0$ and $u(x,0) = 3\sin\frac{\pi x}{l}, 0 < x < l$.

Solution: Given equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \cdots\cdots \quad (1)$$

The boundary conditions are

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \text{for all } t$$

and the initial condition is

$$u(x,0) = 3\sin\frac{\pi x}{l}, \quad 0 < x < l.$$

The required solution of (1) is given by

$$u(x,t) = (A\cos px + B\sin px)e^{-c^2p^2t}. \quad \cdots\cdots \quad (2)$$

Using condition $u(0,t)$ in (2), we get

$$u(0,t) = 0 \Rightarrow Ae^{-c^2p^2t} = 0 \Rightarrow A = 0.$$

(2) reduces to

$$u(x,t) = B\sin pxe^{-c^2p^2t}. \quad \cdots\cdots \quad (3)$$

Applying the condition $u(l,t) = 0$ in (3), we get

$$u(l,t) = B \sin pl e^{-c^2p^2t} = 0$$
$$\Rightarrow \sin pl = 0$$
$$\Rightarrow pl = n\pi, \quad n = 1, 2, \cdots$$
$$\Rightarrow p = \frac{n\pi}{l}, \quad n = 1, 2, \cdots$$

Substituting the value of p in (3), we get

$$u(x,t) = Be^{-c^2n^2\pi^2t/l^2} \sin \frac{n\pi x}{l} = b_n e^{-c^2n^2\pi^2t/l^2} \sin \frac{n\pi x}{l}.$$

$\therefore$ The most general solution of (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-c^2n^2\pi^2t/l^2} \sin \frac{n\pi x}{l} \quad \cdots\cdots \quad (4)$$

Using condition $u(x,0) = 3\sin\frac{\pi x}{l}$ in (4), we get

$$3\sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

Comparing the coefficients of like terms, we get $b_1 = 3$.

Hence from (4), the desired solution is

$$u(x,t) = 3e^{-c^2\pi^2t/l^2} \sin\frac{\pi x}{l}.$$

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